STAT0041: Stochastic Calculus

Lecture 6 - Brownian Motion

Lecturer: Weichen Zhao — Fall 2024

Key concepts:

- *Brownian Motion.*

6.1 Definition of Brownian motion

In history, Einstein gave the physical description of Brownian motion in 1905 [1]. We first give the definition of Brownian motion.

Definition 6.1 (Brownian motion) *Let $(B_t)_{t\geq 0}$ be a stochastic process, if it satisfies:*

(1) Sample path is almost surely continuous;

(2) $\forall 0 < t_1 < t_2 < \cdots < t_n$, $B_0, B_{t_1} - B_0, \cdots, B_{t_n} - B_{t_{n-1}}$ are mutually independent;

(3) $B_{t+s} - B_t$ is Gaussian distributed with $N(0, s)$.

We say $(B_t)_{t\geq 0}$ is a (one-dimension) ***Brownian motion****. If $B_0 = 0$, we say $(B_t)_{t\geq 0}$ is a standard Brownian motion.*

Consider natural filtration of $(B_t)_{t\geq 0}$

$$\mathscr{F}_t := \sigma(\{B_s : 0 \leq s \leq t\}).$$

Condition (2) in definition 6.1 is equivalent to $\forall\, 0 \leq s < t$, $B_t - B_s$ and $\mathscr{F}_s$ are independent.

Remark 6.2 *A stochastic process X_t is called a Gaussian process, if for all $0 < t_1 < t_2 < \cdots < t_n$ the vector $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian random vector. Brownian motion is a Gaussian process.*

Remark 6.3 *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, then $Cov(B_t, B_s) = t \wedge s$.*

Following remark give a useful equivalent definition of Brownian motion.

Remark 6.4 *Let $B_0 = 0$, a stochastic process with continuous sample path $(B_t)_{t\geq 0}$ is a Brownian motion if and only if $(B_t)_{t\geq 0}$ is a Gaussian process, and*

$$\mathbb{E}B_t = 0,\ \mathbb{E}[B_t B_s] = t \wedge s, \quad \forall\, t, s$$

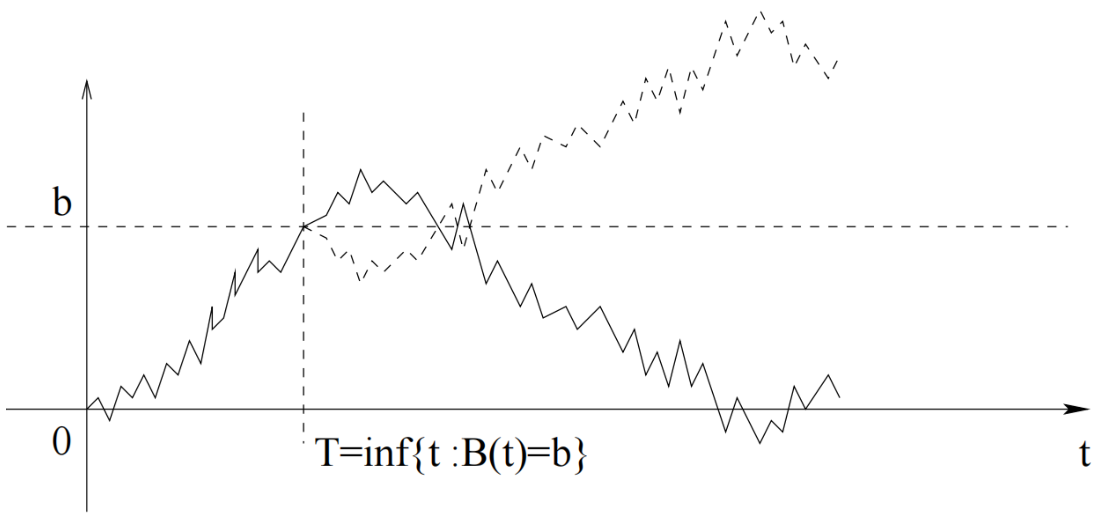

Figure 6.1: The reflection principle in the case of the first hitting time of level b. [2]

Construction of Brownian motion. Refer to section 1.1.2 of [2]

6.2 Basic properties of Brownian motion

Proposition 6.5 *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, then*

(1) (shifting) *Fixed $t_0 \geq 0$, $B_{t_0+t} - B_{t_0}$ is a Brownian motion;*

(2) (scaling) *$\forall c \neq 0 \in \mathbb{R}$, $cB_{\frac{t}{c^2}}$ is a Brownian motion. Especially, $c = -1$, $-B_t$ is a Brownian motion;*

(3) (time reversal)

$$\tilde{B}_t = \begin{cases} 0, & t = 0 \\ tB_{\frac{1}{t}}, & t > 0 \end{cases}$$

is a Brownian motion.

Corollary 6.6 (Law of large numbers) *Almost surely,* $\lim_{t\to\infty} \frac{B_t}{t} = 0$.

Lemma 6.7 (Reflection principle) *If T is a stopping time and $\{B(t): t \geqslant 0\}$ is a standard Brownian motion, then the process $\{B^*(t): t \geqslant 0\}$ called* ***Brownian motion reflected*** *at T and defined by*

$$B^*(t) = B(t)1_{\{t\leqslant T\}} + (2B(T) - B(t))1_{\{t>T\}}$$

is also a standard Brownian motion.

The proof of reflection principle need strong Markov property of Brownian motion. See details in section 2.2.1 of [2].

References

[1] Einstein Albert. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. Annalen der physik 4 (1905).

[2] Mörters Peter, and Yuval Peres. Brownian motion. Vol. 30. Cambridge University Press, 2010.